

**Indian Statistical Institute, Bangalore**

B. Math (Hons.) Second Year

First Semester - Optimization

Mid-Term Exam Duration : 3 hours Max Marks 40 Date : September 15, 2017

**Answer any five, each question carries 8 marks, total marks: 40**

1. (a) Prove that pivot columns in an upper echelon matrix are linearly independent (*Marks: 4*).  
(b) Let  $A$  be a  $p \times q$  upper echelon matrix with  $k$  pivots. If  $k \leq p < q$ , prove that  $A$  is not one-one.

2. (a) Find the  $QR$  decomposition of  $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$  (*Marks: 4*).

(b) Give a  $3 \times 3$  matrix to show that  $QR$  decomposition is not unique.

3. (a) Prove that every  $p \times q$ -matrix has singular value decomposition.  
(b) Prove that the singular values of  $A$  and  $A^*$  coincide. (*Marks: 3*).
4. (a) Let  $A \in M_{p \times q}(\mathbb{C})$  and  $s_1$  be the first singular value. Prove that  $s_1 = \|A\|$ .  
(b) If rank of a  $p \times q$ -matrix  $A$  is  $q$ , prove that  $s_q = \min_{\|x\|=1} \|Ax\|$  (*Marks: 3*).
5. Let  $S$  be a subspace of  $\mathbb{R}^n$  and  $a, u \in \mathbb{R}^n$ . Denote  $W = a + S = \{a + x \mid x \in S\}$ .  
(a) Prove that  $u$  can be written uniquely as  $w + y$  for  $w \in W$  and  $y \in S^\perp$ .  
(b) Prove that  $\min_{y \in W} \|u - y\|$  has a unique solution (*Marks: 4*).
6. (a) If  $A$  is a non-negative  $n \times n$ -matrix, prove that  $(I + A)^{n-1}$  is positive.  
(b) Let  $T$  and  $S$  be non-negative matrices such that  $T$  is irreducible and  $T - S$  is non-negative. Prove that  $\text{spr}(T) \geq \text{spr}(S)$  and the equality occurs only if  $T = S$  (*Marks: 5*).
7. (a) Let  $A$  be a non-negative irreducible matrix. If  $\lambda$  is an eigenvalue of  $A$  with eigenvector  $x \geq 0$ , prove that  $\lambda$  is the spectral radius of  $A$  (*Marks: 4*).

- (b) Prove that  $A = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$  is irreducible and find its Perron-pair .